Statistical Tolerance Bounds: Overview and Applications to Space Systems

James Womack
The Aerospace Corporation

3 May 2011

© The Aerospace Corporation 2011

Statistical Intervals

- Decisions are frequently made based on limited sample data
  - *Examples: Strength of a material must be at least X*

    Minimum strength of a material must exceed max load applied

    Determine minimum strength of a material over several environments
- Sample data can be used to estimate mean strength or probability of exceeding limits, but provides no information about the precision of the estimates
  - *There may be big differences between the estimated values and what the true values are, if unlimited data were available*
- Statistical intervals quantify the uncertainty associated with an estimate

James.M.Womack@aero.org

AEROSPACE

Types of Statistical Intervals

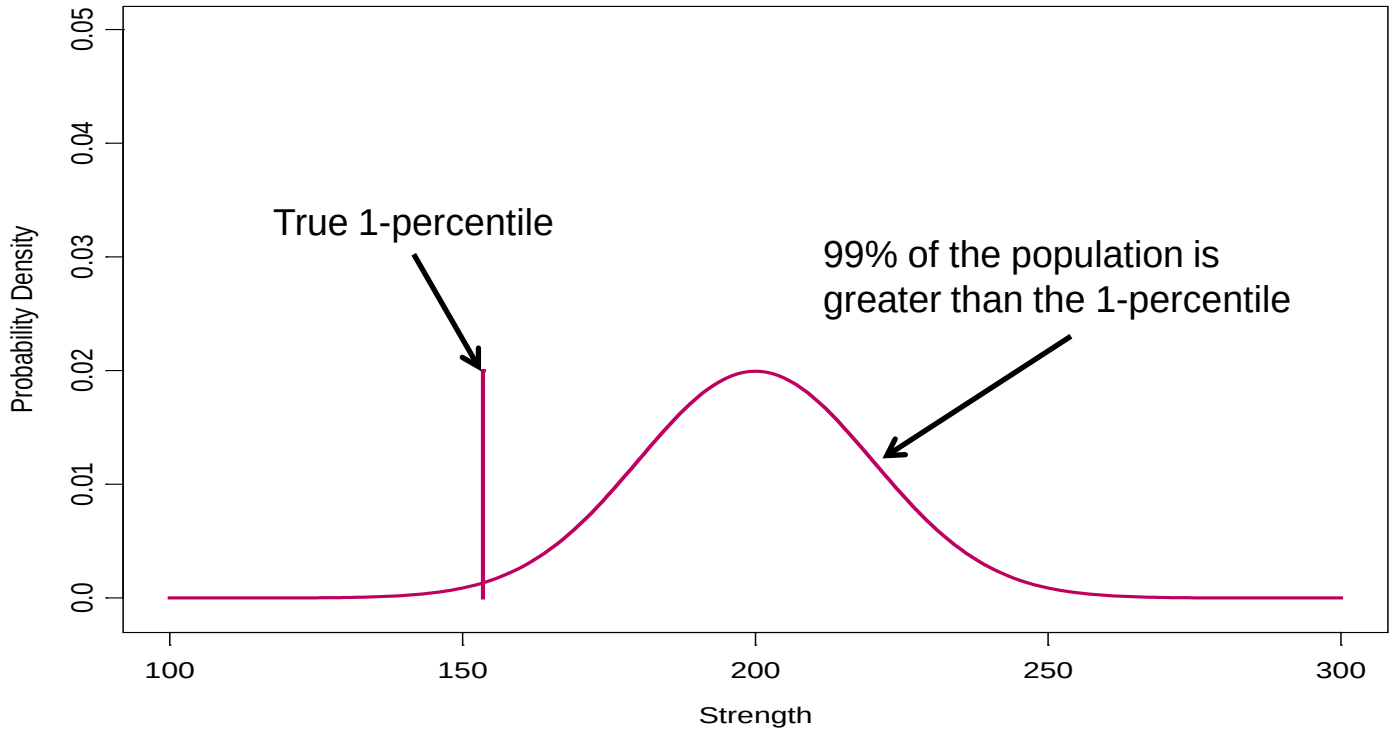
- Appropriate type statistical interval depends on the application
  - *<u>Confidence and Tolerance Intervals</u> are for describing the population or process from which the sample data has been selected*
  - *<u>Prediction Intervals</u> are for predicting the results of a future sample*
- Confidence intervals: enclose means, variances and other population parameters
- Tolerance intervals are used to contain a specified proportion of a population
  - *Lower tolerance bound for bounding the population from below (minimum strength)*
  - *Upper tolerance bound for bounding the population from above (maximum load)*
  - *Tolerance interval for enclosing the population both above and below*
- Example: If T is an upper tolerance bound with 90% confidence for .99 of the population, then we are 90% confident that 99% of the population is less than T
  - *T is denoted as a (.99, .90), where 99 represents the proportion of the population bounded and 90 represents the confidence level*

James.M.Womack@aero.org

AEROSPACE

1-Percentile of Normal Distribution

- Assume the population is normal
  - *Mean (μ = 200) and variance (σ^2 = 400)*
- Then the 1-percentile = $\mu - 2.326\sigma = 153.5$



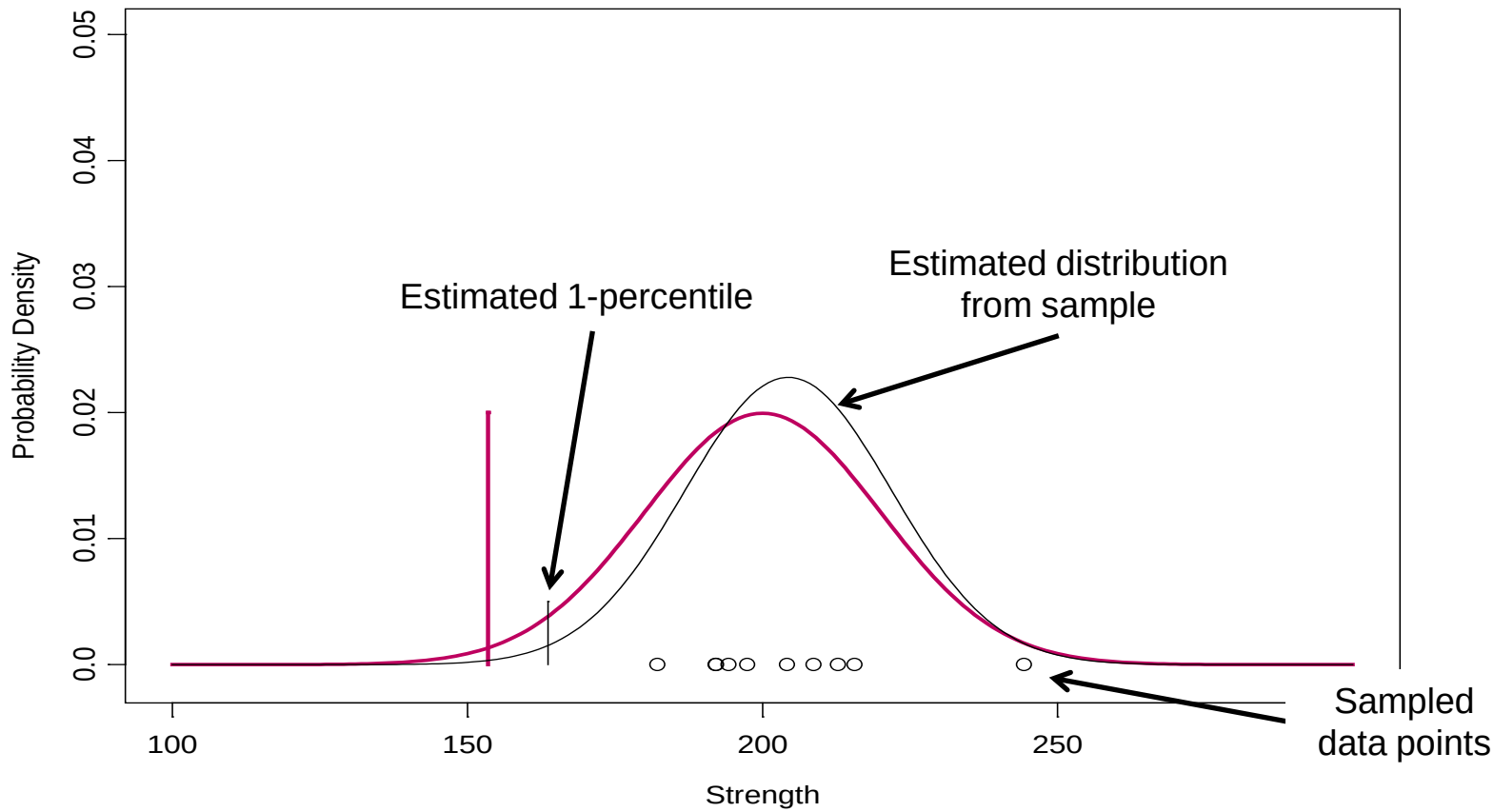
- When the mean and variance are both unknown, we must sample from this distribution to estimate the 1-percentile

James.M.Womack@aero.org

AEROSPACE

Sample of 10 Measurements Used to Estimate 1-Percentile

- Use sample mean, $\bar{x}$, in place of μ and sample variance, s^2, in place of σ^2
- Estimated 1-percentile = $\bar{x} - 2.326s$

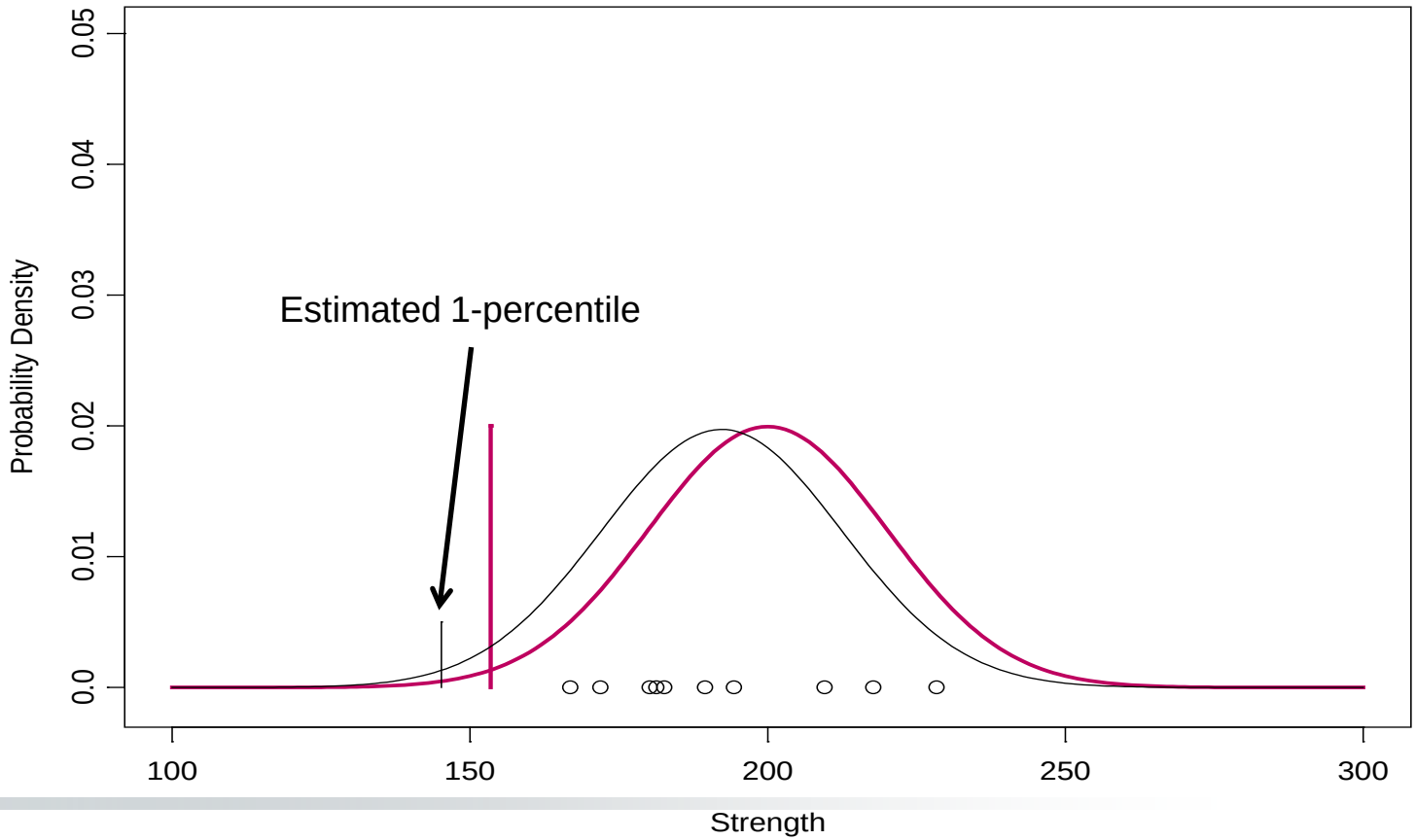


James.M.Womack@aero.org

AEROSPACE

Another Sample of 10 Measurements Used to Estimate 1-Percentile

- Use sample mean, $\bar{x}$, in place of μ and sample variance, s^2, in place of σ^2
- Estimated 1-percentile = $\bar{x} - 2.326s$

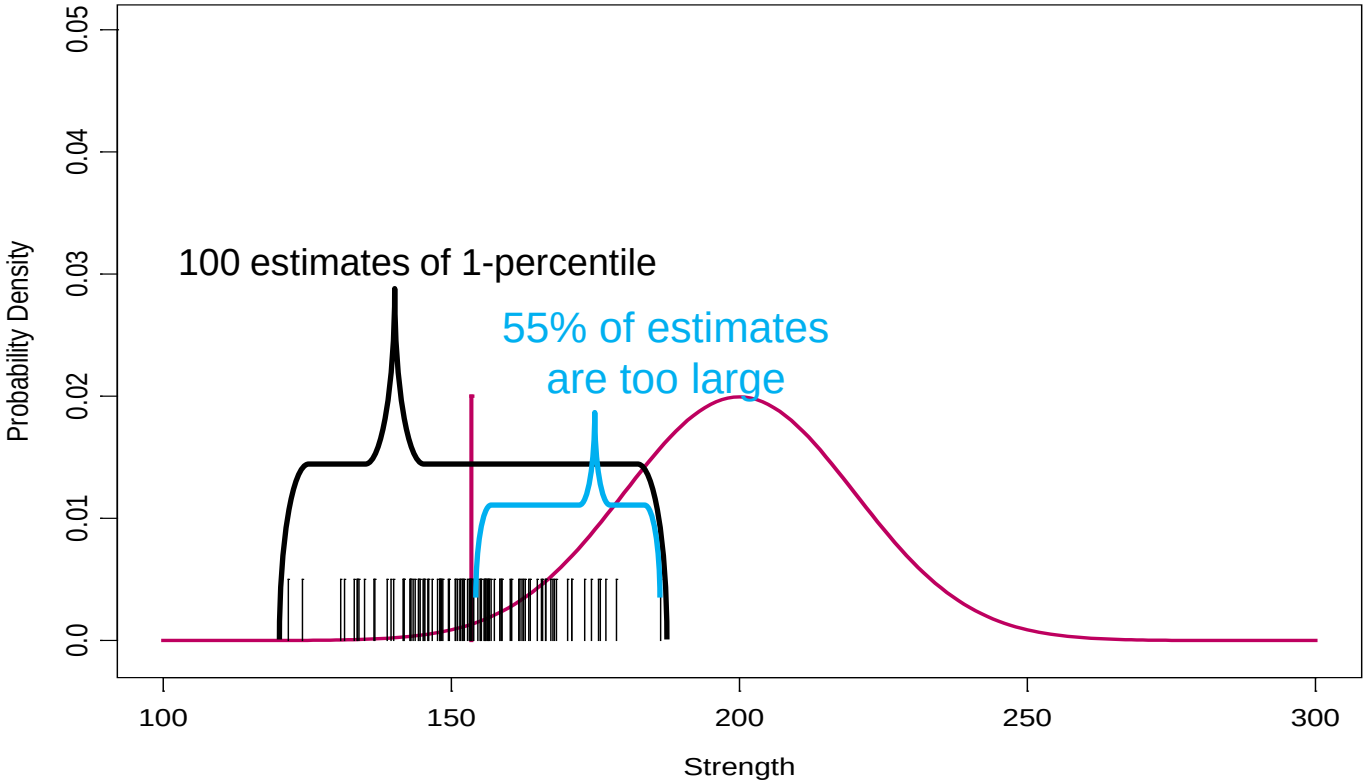


James.M.Womack@aero.org

AEROSPACE

100 Samples of 10 Measurements Used to Estimate 99-percentile

- Use sample mean, $\bar{x}$, in place of μ and sample variance, s^2, in place of σ^2
- Estimated 1-percentile = $\bar{x} - 2.326s$



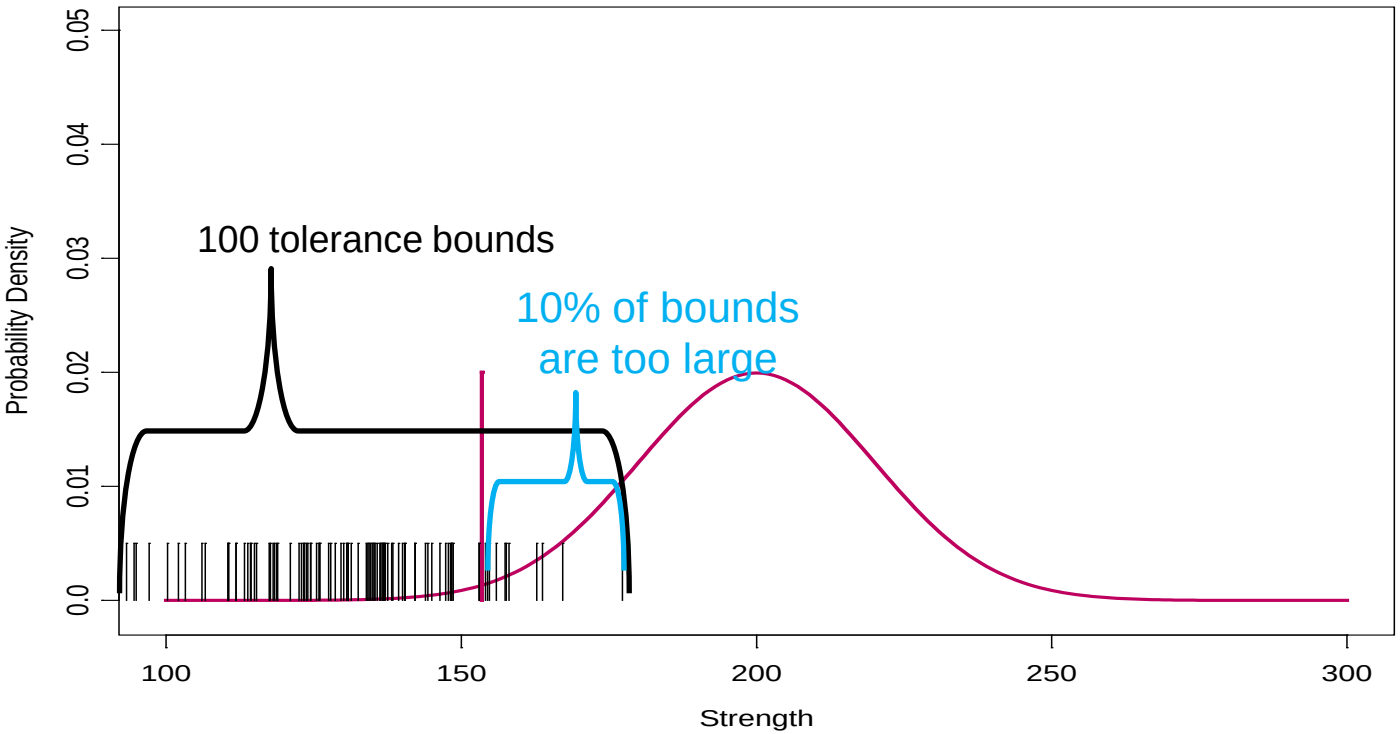
- A (.99,.90) lower tolerance bound is of the form: $\bar{x} - ks$
  - *k is selected so that 90% of the samples will produce a bound that encloses 99 percent of the population (k = 3.532 for sample size of 10)*

James.M.Womack@aero.org

AEROSPACE

(.01,.90) Lower Tolerance Bounds: 100 Samples of 10 Measurements

- Use sample mean, $\bar{x}$, in place of μ and sample variance, s^2, in place of σ^2
- (.99,.90) Lower tolerance bound = $\bar{x} - 3.532s$



- Increasing sample size will decrease spread of tolerance bounds
- Increasing confidence level will decrease percent of time tolerance bound is too large

James.M.Womack@aero.org

AEROSPACE

Tolerance Bounds for Normal Distribution

- $X_1, X_2, \ldots, X_n$ sample from normal population, $N(\mu,\sigma^2)$, with unknown μ and σ^2
  - *Let z_p be the p quantile of standard normal distribution*
  - *p quantile of $N(\mu,\sigma^2)$ is $q_p = \mu + z_p \sigma$*
- The 1-α upper confidence bound for q_p is a (p,1- α) one-sided upper tolerance bound for $N(\mu,\sigma^2)$
  - *Find k such that* $P_{\bar{X},S}\left\{P\left(X < \bar{X} + kS \mid \bar{X}, S\right) > p\right\} = 1-\alpha$
  - *Or* $P\left\{\dfrac{\dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}} + z_p\sqrt{n}}{\dfrac{S}{\sigma}} > -k\sqrt{n}\right\} = 1-\alpha$ *(Non-central t distribution)*
  - *k* = $\dfrac{1}{\sqrt{n}} t_{n-1;1-\alpha}(z_p\sqrt{n})$
- The (p,1- α) upper tolerance bound is $\bar{X} + t_{n-1;1-\alpha}(z_p\sqrt{n})\dfrac{S}{\sqrt{n}}$
- The (p,1- α) lower tolerance bound is $\bar{X} - t_{n-1;1-\alpha}(z_p\sqrt{n})\dfrac{S}{\sqrt{n}}$

James.M.Womack@aero.org

AEROSPACE

Splus Code for Normal Tolerance Bound

```
# Computes (p,1-alpha) upper tolerance bound on X ~ Normal
#
# Inputs: xdata (vector of sample data), p = percentile, alpha = 1 - confidence level

library(envstats)              # library for non-central t

UpperToleranceBound <- function(p,alpha,xdata){
        n <- length(xdata)
        Ssq <- var(xdata)
        UTB <- mean(xdata) + qt(1-alpha, n-1, ncp = qnorm(p)*sqrt(n))*sqrt(Ssq/n)
        UTB
}
```

```
# Example
> xdata <- c(1.822938, 1.143871, 0.972309, -0.078231, 0.480773, 0.710025, -0.573717,
0.272126, 0.016359, -0.596675)
> UpperToleranceBound(.99,.10,xdata)
[1] 3.1371
```


James.M.Womack@aero.org

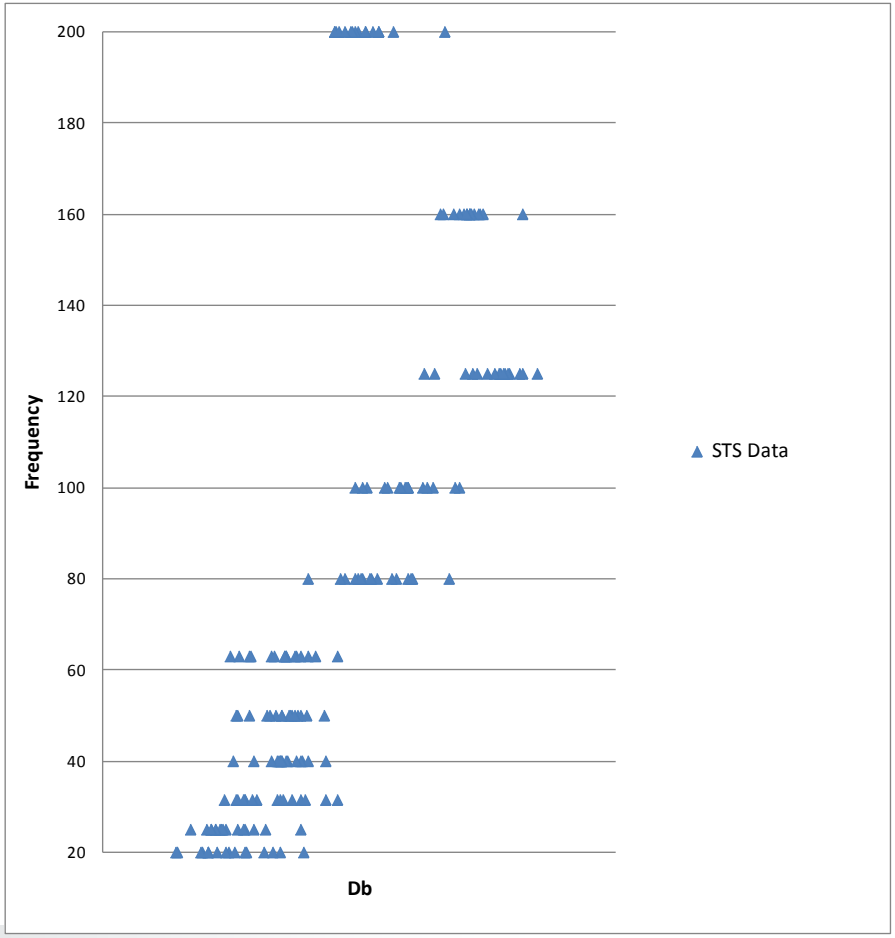
AEROSPACE

Orbiter Heat Shield Main Engine Ignition Acoustic Environment

Upper Tolerance Bound Example

- After post flight analysis, 16 flights remaining with data judged suitable for use in updating base heat shield environment limits
- MIL-STD- 1540E and NASA STD –HDBK-7005 approach for this limit:
  - (.99,.90) upper tolerance bound used for qualification testing
  - (.95,.50) upper tolerance bound used for acceptance test

- STS acoustic vibration dataset:



James.M.Womack@aero.org

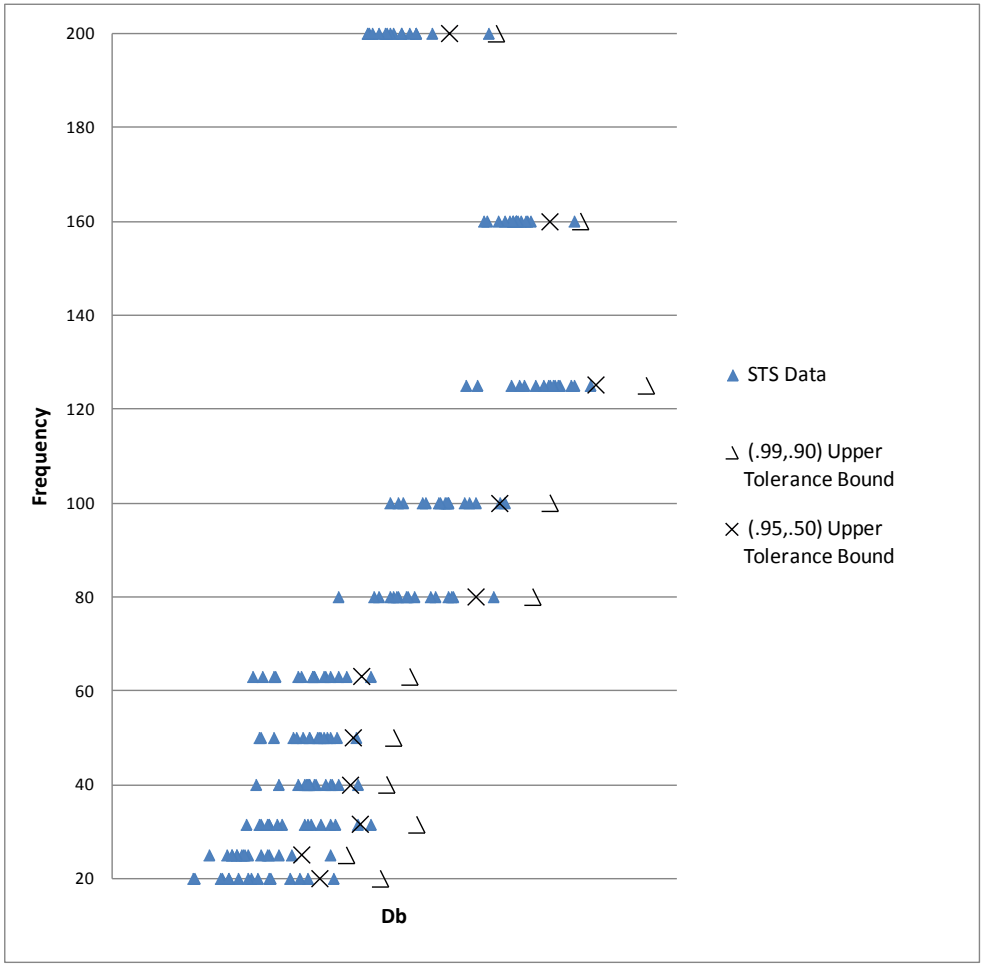
AEROSPACE

Orbiter Heat Shield Main Engine Ignition Acoustic Environment

Upper Tolerance Bound Example (2)

- Assume independence between frequency bands
- Probability plots show data can be reasonably modeled by normal distribution
- Normal upper tolerance bounds:
  - *(.99,.90):* $\bar{x} + 3.172s$
  - *(.95,.50):* $\bar{x} + 1.678s$

- Results



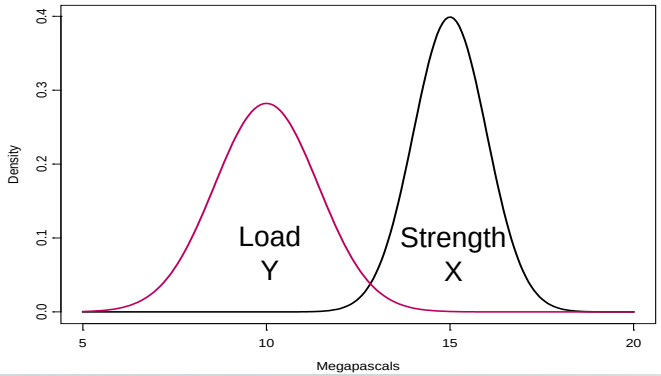
James.M.Womack@aero.org

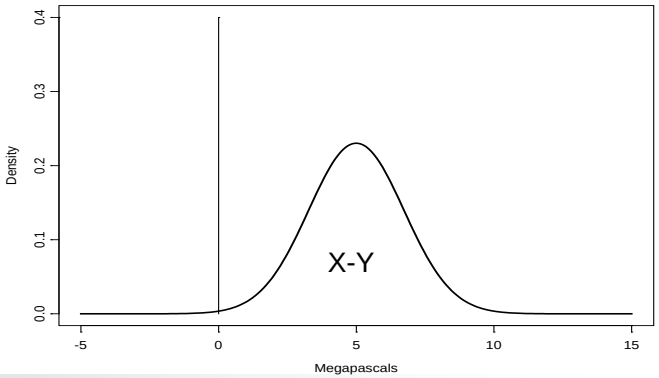
AEROSPACE

CEV Parachute Assembly System (CPAS) Reliability

Load-Strength Reliability Problem

- To calculate the reliability for the main and drogue parachute components, both the strength and load distributions are characterized by independent normal distributions
- Reliability, *R*, is the probability that the strength, *X*, is greater then the load, *Y*, i.e. P(*X*-*Y* > 0)
  - *Distribution of X-Y is normal with mean* $\mu_X - \mu_Y$ *, and variance* $\sigma_X^2 + \sigma_Y^2$
  - *R* = $1 - \Phi\left(\frac{\mu_Y - \mu_X}{\sqrt{\sigma_X^2 + \sigma_Y^2}}\right)$*, where* Φ *is the cumulative normal distribution*
  - *Since the means and variances are unknown, we can only estimate the reliability so we also need a lower confidence bound on R*
- If we select a quantile, p, such that the (p,1-α) lower tolerance bound = 0, then p is a 100(1- α)% lower confidence bound on R





James.M.Womack@aero.org

AEROSPACE

Lower Tolerance Bounds for X-Y

- $X_1, X_2, \ldots X_{n_1}$ sample from normal population, $N(\mu_X, \sigma_X^2)$,
- $Y_1, Y_2, \ldots Y_{n_2}$ sample from normal population, $N(\mu_Y, \sigma_Y^2)$,
- 1-p quantile is: $L_p = \mu_X - \mu_Y - z_p\sqrt{\sigma_X^2 + \sigma_Y^2}$
- The (p,1-α) lower tolerance bound is:
  - *Exact solution if ratio of variances is known (Hall):*

$$\overline{X} - \overline{Y} - t_{n1+n2-2,1-\alpha}(z_p\sqrt{\nu})\frac{S_d}{\sqrt{\nu}}$$

$$\text{where } \nu = \frac{n_1(1+q)}{q + n_1/n_2}, \quad S_d^2 = \frac{\left(1+\tfrac{1}{q}\right)\left[(n_1-1)S_X^2 + (n_2-1)qS_Y^2\right]}{n_1+n_2-2}, \text{ and } q = \frac{S_X^2}{S_Y^2}$$

  - *Approximate solution for unknown and arbitrary variances (Guo-Krishnamoorthy):*

$$\min\left(\overline{X} - \overline{Y} - t_{f_i,1-\alpha}(z_p\sqrt{\nu_i})\sqrt{\frac{S_X^2 + S_Y^2}{\nu_i}},\ i = 1,2\right)$$

$$\text{where } \nu_1 = \frac{n_1(1+q_1)}{q_1 + n_1/n_2}, \quad \nu_2 = \frac{n_2(1+q_2)}{q_2 + n_2/n_1}, \quad f_1 = \frac{(n_1-1)(q_1+1)^2}{q_1^2 + (n_1-1)/(n_2-1)}, \quad f_2 = \frac{(n_2-1)(q_2+1)^2}{q_2^2 + (n_2-1)/(n_1-1)}, \quad q_1 = \frac{S_X^2(n_2-3)}{S_Y^2(n_2-1)}, \quad q_2 = \frac{S_Y^2(n_1-3)}{S_X^2(n_1-1)}$$

James.M.Womack@aero.org

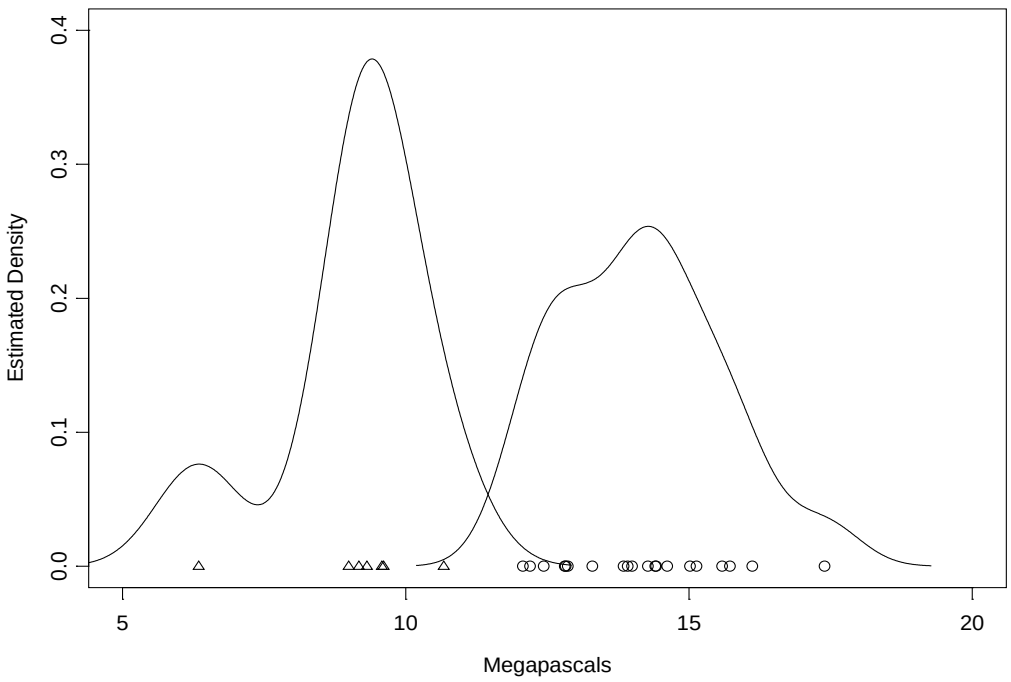
AEROSPACE

CEV Parachute Assembly System (CPAS) Reliability

Load-Strength Reliability Problem (2)

- Simulated data: 20 strength, 7 load measurements:

Strength	16.1	17.4	14.6	12.8	14.0	15.0	14.3	12.9	15.7	14.4	15.6	13.3	13.8	13.9	12.2	12.4	12.8	14.4	15.1
Load	9.3	9.2	6.3	9.6	9.0	9.6	10.7												



- Approximate (p,.90) Lower Tolerance bounds

p	(p,.90) LTB
.99	-1.25
.98	-0.57
.97	-0.14
.9656	0.00
.96	0.17

- 90% confident that P(strength > load) > .9656

James.M.Womack@aero.org

AEROSPACE

Splus Code for Lower Tolerance Bound on X-Y

Ratio of Variances Known

```
# Computes (p,1-alpha) lower tolerance bound on X-Y for known ratio of variances
# inputs: xdata, ydata both vectors, p = percentile, alpha = 1 - confidence level, q1 = (sigma1/sigma2)^2

library(envstats)          # library for non-central t

Tolerance2distsKnownRatio <- function(p,alpha,q1,xdata,ydata)
{
        n1 <- length(xdata)
        n2 <- length(ydata)
        Sdsq <- (1+1/q1)*((n1-1)*var(xdata)+(n2-1)*q1*var(ydata))/(n1+n2-2)
        v1 <- n1*(1+q1)/(q1+n1/n2)
        LTB <- mean(xdata)-mean(ydata)-qt(1-alpha,n1+n2-2, ncp = qnorm(p)*sqrt(v1))*sqrt(Sdsq/v1)
        LTB
}
```


James.M.Womack@aero.org

AEROSPACE

Splus Code for Lower Tolerance Bound on X-Y

Unknown Arbitrary Variances

```
# Computes (p,1-alpha) lower tolerance bound on X-Y for unknown variances
# inputs: xdata, ydata both vectors, p = percentile, alpha = 1 - confidence level

library(envstats)          # library for non-central t

Tolerance2distsUnknownRatio <- function(p,alpha,xdata,ydata)
{
        n1 <- length(xdata)
        n2 <- length(ydata)
        S1sq <- var(xdata)
        S2sq <- var(ydata)
        q1 <- S1sq*(n2-3)/S2sq/(n2-1)
        q2 <- S2sq*(n1-3)/S1sq/(n1-1)
        v1 <- n1*(1+q1)/(q1+n1/n2)
        v2 <- n2*(1+q2)/(q2+n2/n1)
        f1 <- (n1-1)*(q1+1)^2/(q1^2+(n1-1)/(n2-1))
        f2 <- (n2-1)*(q2+1)^2/(q2^2+(n2-1)/(n1-1))
        LTB1 <- mean(xdata)-mean(ydata)-qt(1-alpha,f1, ncp = qnorm(p)*sqrt(v1))*sqrt((S1sq+S2sq)/v1)
        LTB2 <- mean(xdata)-mean(ydata)-qt(1-alpha,f2, ncp = qnorm(p)*sqrt(v2))*sqrt((S1sq+S2sq)/v2)
        LTB <- min(LTB1,LTB2)
}
```

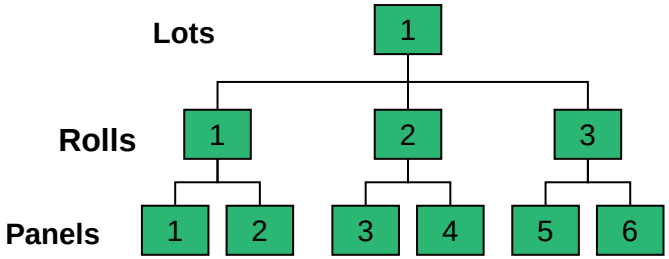

James.M.Womack@aero.org

AEROSPACE

Structural Allowable for Silica Cloth Phenolic

Structured Data Example

- Silica Cloth Phenolic (SCP) is used as part of the RSRM and RSRM-V nozzles (TPS for flex bearing)
- Allowable stress (working stress or design allowable) for a material is the maximum stress at which one can be reasonably certain that failure will not occur
- Two one-sided tolerance limits are used
  - *A-basis: (.99,.95) lower tolerance bound on material strength*
  - *B-basis: (.90,.95) lower tolerance bound on material strength*
- Strength of composite materials may have several sources of variation
  - *Lot, Rolls, Panel*
  - *Temperature*
  - *Within panel variation*
  - *Other sources*



- Data with multiple sources of variability should be tested to determine if data is structured

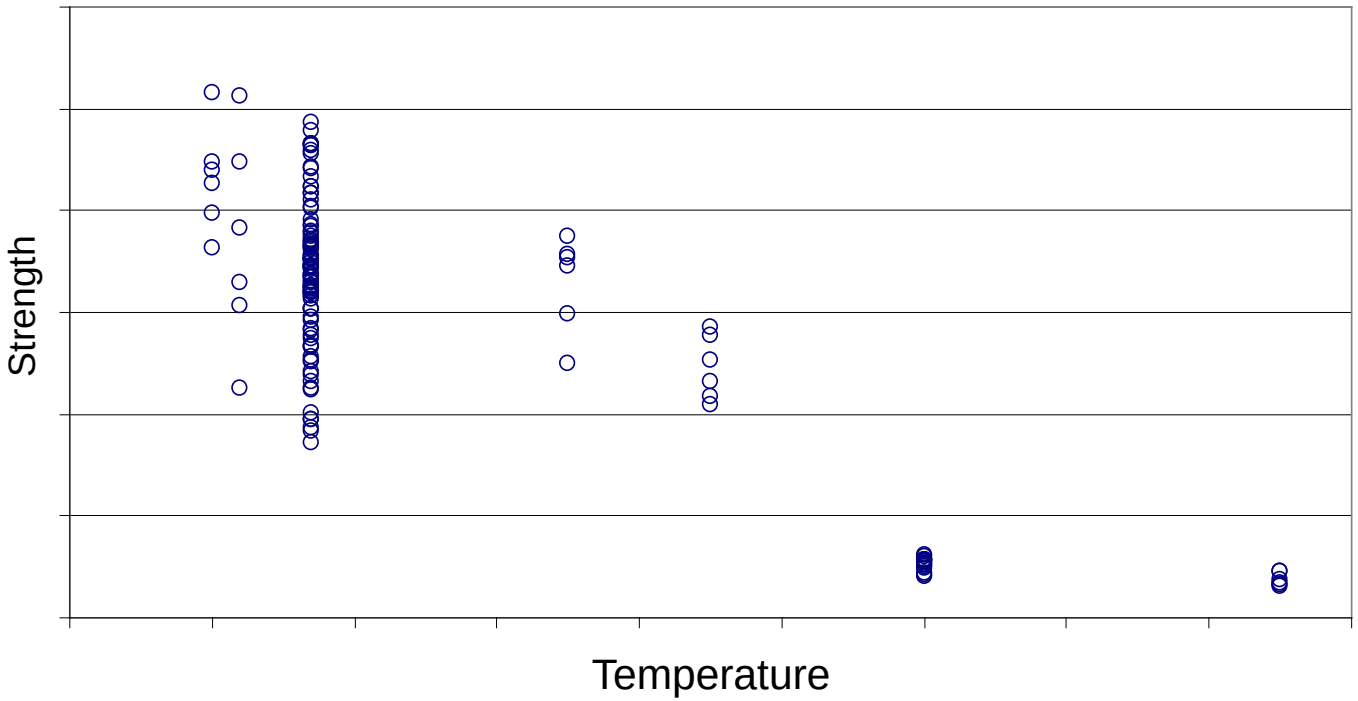
James.M.Womack@aero.org

AEROSPACE

Structural Allowable for Silica Cloth Phenolic

Structured Data Example (2)

- Silica Cloth Phenolic Panel Across-Ply Tensile Strength Data



- Objective: Determine lower tolerance bound at each temperature

James.M.Womack@aero.org

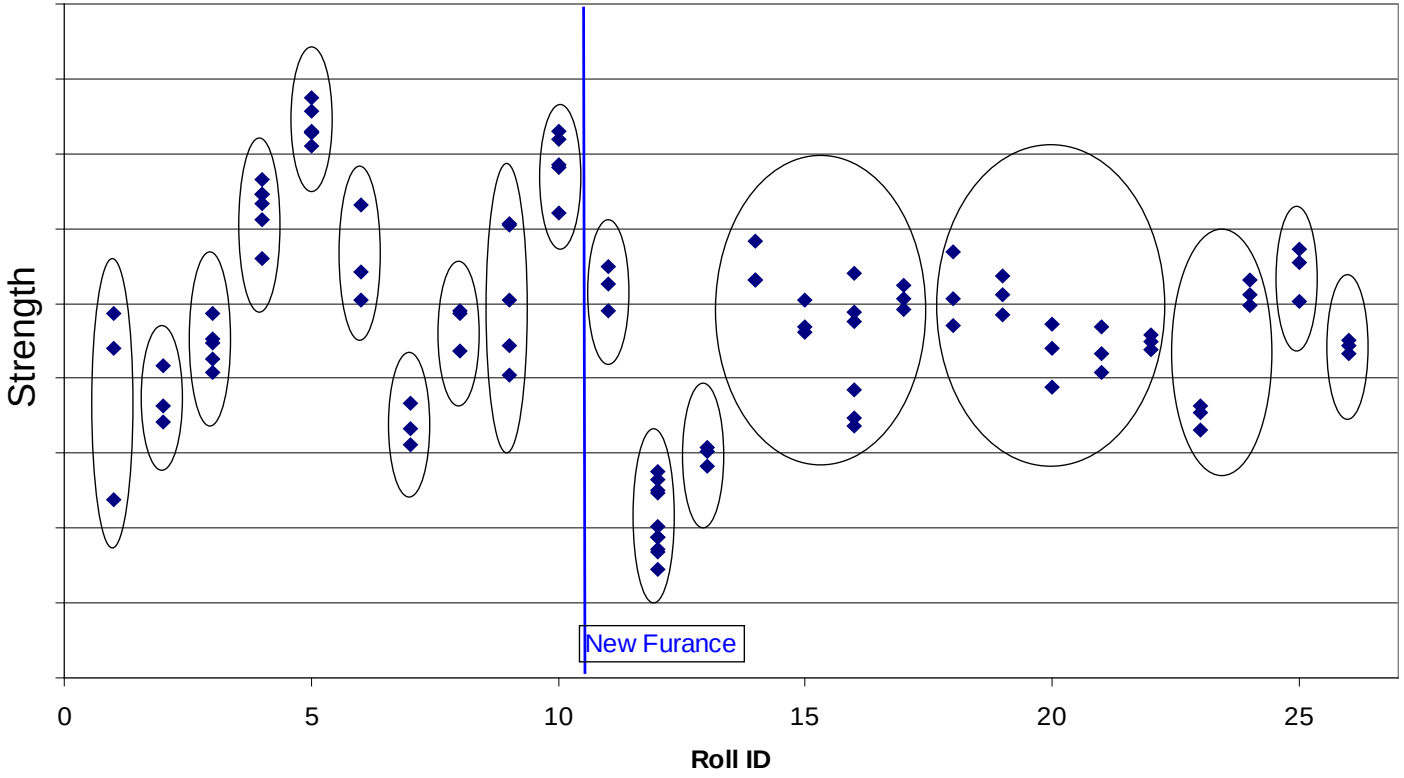
AEROSPACE

Structural Allowable for Silica Cloth Phenolic

Structured Data Example (3)

Silica Cloth Phenolic Panel Across-Ply Tensile Strength Data

Single Test Temperature



Anderson-Darling k-sample test proves the data is structured

James.M.Womack@aero.org

AEROSPACE

Estimating Data with 70 F Data Structure

Lot ID	Roll ID	Samples
3	3	5
4	4	6
5	5	6
6	6	3
7	7	3
8	8	3
9	9	5
10	10	5
11	11	3
12	12	10
13	13	3
14	14	3
	15	3
	16	6
	17	3
15	18	3
	19	3
	20	3
	21	3
	22	3
16	23	3
	24	3
17	25	3
18	26	3

- Randomly draw 16 lot means from $N(\mu,\sigma_{lot})$
- Randomly draw 24 roll means from $N(0, \sigma_{roll|lot})$
- Randomly draw 94 residual values from $N(0, \sigma_e)$
- Combine random data into lot, roll, sample data structure

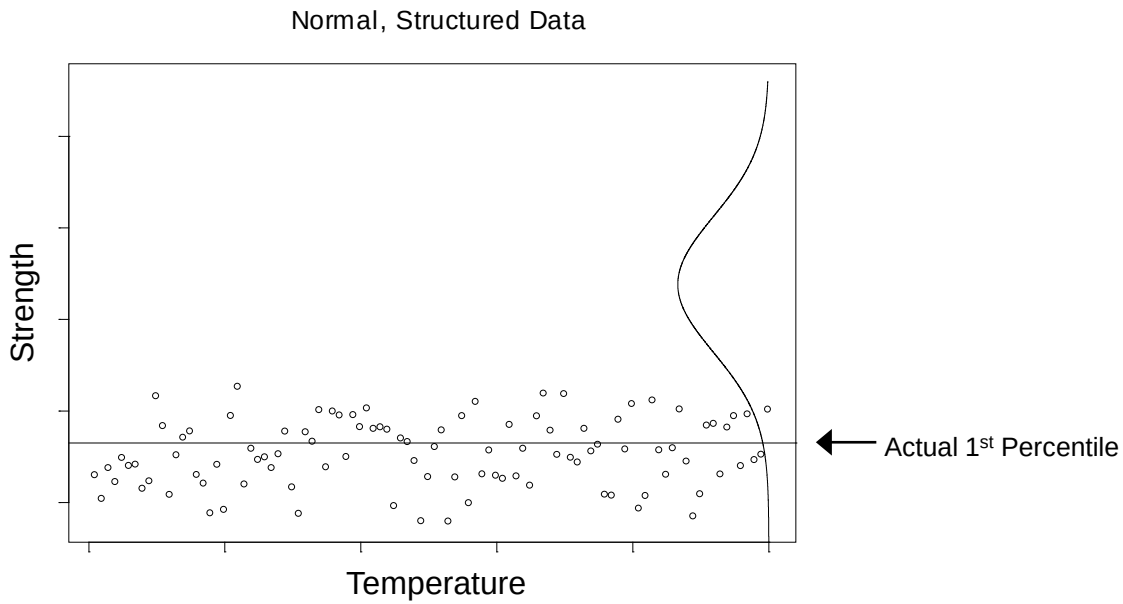
James.M.Womack@aero.org

AEROSPACE

Structural Allowable for Silica Cloth Phenolic

Structured Data Example (4)

- (.99,.95) normal tolerance bounds applied to structured data
  - *Using the estimated structure as in the SCP Tensile Strength data, 100 samples of size 94 are drawn and the tolerance bound is calculated*
  - *Simulated data includes variability between lot, within lots, within roll/panel*



- Normal tolerance bound is an overestimate approximately 35% of the time, not effective for structured data

James.M.Womack@aero.org

AEROSPACE

Mixed Effects Model For Silica Cloth Phenolic

Structured Data Example (5)

- Strength(T) = $\beta_0 + \beta_1 T + L + R + \varepsilon_k$ [Normal($\beta_0 + \beta_1 T$, $\sqrt{\sigma_{lot}^2 + \sigma_{roll|lot}^2 + \sigma_e^2}$)]
  - *T = temperature of SCP*
  - β_0, β_1 = slope and intercept parameters
  - *L = random lot effects, Normal(0,* σ_{lot} *)*
  - *R = random roll effect within a lot, Normal(0,* $\sigma_{roll|lot}$*)*
  - ε_k *= Normal(0,* σ_e *)*
  - *Panel-to-panel variability ignored (only lot 14 roll 16 has more than 1 panel)*

- 1-Percentile of strength = $\beta_0 + \beta_1 T - 2.326 \sqrt{\sigma_{lot}^2 + \sigma_{roll|lot}^2 + \sigma_e^2}$

- (.99,.95) lower tolerance bound is a 95% lower confidence bound on 1-percentile
  - *Methods for computing tolerance bounds for this type of structured data are currently not available (unbalanced data with more than two variance terms)*
  - *Methods exist for unbalanced data with two variance terms and balanced data with more than two variance terms*

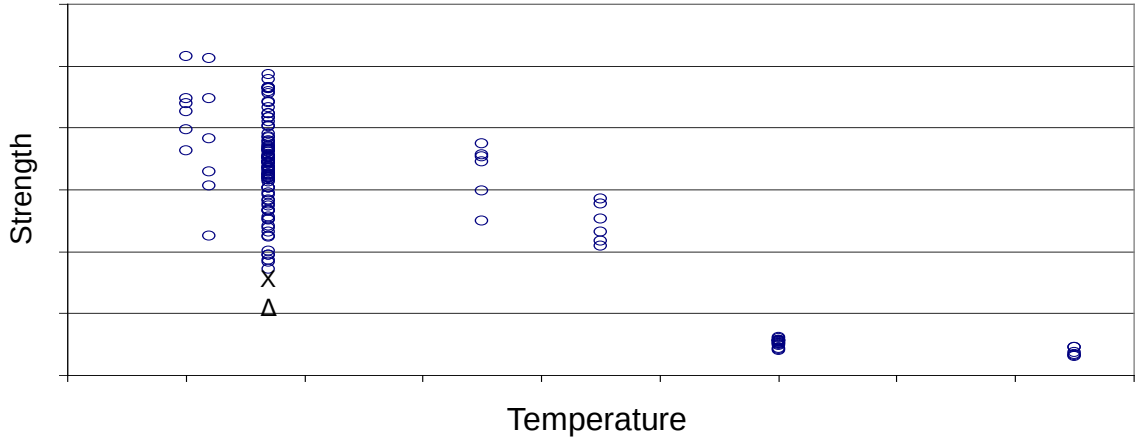
James.M.Womack@aero.org

AEROSPACE

Structural Allowable for Silica Cloth Phenolic

Structured Data Example (6)

- Silica Cloth Phenolic Panel Across-Ply Tensile Strength Data at 70 C
  - *Model as a mixed model with one random effect and unbalanced data*
    - Temperature is fixed effect
    - Combine lot and roll effect into single effect
    - Strength(T) = $\beta_0 + \beta_1 T + L + \varepsilon_k$ [Normal($\beta_0 + \beta_1 T, \sqrt{\sigma_{lot}^2 + \sigma_e^2}$)]
- Approximate methods exist for this case
  - *RECIPE ANOVA model used to compute (.99,.95) lower tolerance bound*
    - http://www.itl.nist.gov/div898/software/recipe/
  - *Splus code (based on Krishnamoorthy & Mathew)*



James.M.Womack@aero.org

AEROSPACE

Splus Code for Balanced Mixed Effects Models

```
# Computes Lower Tolerance Bounds for a Balanced Nested Design
# 1 Fixed Factor (Temperature) and 1 Random Factor (lot)
# Temperature has a levels and lot has b levels and n is the number of replicates
# Observed data is an array Y(i,j,el), i=1,2,...a, j=1,2,...,b, el=1,2,...,n
# p = the percentile of the tolerance bound
# alpha = 1 - confidence level of the tolerance bound

LTBmixedUnbalanced <- function(p,alpha,Y){
        a <- dim(Y)[1]
        b <- dim(Y)[2]
        n <- dim(Y)[3]
        f1 <- a*(b-1)
        f2 <- a*b*(n-1)
        d1 <- 1/(n*b)
        d2 <- (b-1)/(n*b)
        c1 <- 1/n
        c2 <- 1 - 1/n
        LTB <- rep(0,a)
        for (temp in 1:a){
                theta_hat <- sum(Y[temp,,])/(b*n)  # Step 1
                ss1 <- 0
                for (i in 1:a){
                        for(j in 1:b){
                                ss1 <- ss1 + ( sum(Y[i,j,])/n - sum(Y[i,,])/(b*n) )^2
                        }
                }
                ss2 <- 0
                for (i in 1:a){
                        for(j in 1:b){
                                for (el in 1:n){
                                        ss2 <- ss2 + ( Y[i,j,el] - sum(Y[i,j,])/n )^2
                                }
                        }
                }
                k <- 100000  # Step 2
                Z <- rnorm(k)  # Step 3
                U1 <- rchisq(k,f1)
                U2 <- rchisq(k,f2)
                G1 <- ss1/U1  # Step 4
                G2 <- ss2/U2
                Gtheta <- theta_hat - Z*sqrt(d1*G1 + d2*G2)
                G8 <- Gtheta - qnorm(p)*sqrt(c1*G1 + c2*G2)  # Step 5
                LTB[temp] <- quantile(G8,alpha)
        }
        LTB
}
```


James.M.Womack@aero.org

AEROSPACE

Summary

- Tolerance bounds are commonly used to verify requirements on space systems
  - *Determines limits of acceptability*
  - *Can also be used to determine required sample sizes*
  - *Includes uncertainty due to small sample sizes*

- Computing tolerance bounds
  - *First check distribution of data*
    - There are tolerance bounds for other types of distributions
  - *Check for structured data*
  - *Apply correct method*

- Research area: Tolerance bounds for general mixed effects or random effects models with unbalanced data

James.M.Womack@aero.org

References

- Hall, I. J. (1984), “Approximate one-sided tolerance limits for the difference of sum of two independent normal variates,” Journal of Quality Technology, 11, 13-19.
- Guo, H., Krishnamoorthy, K. (2004), “New Approximate Inferential Methods for the Reliability Parameter in a Stress-strength Model: The Normal Case,” Communications in Statistics - Theory and Methods, 33, 1715-1731.
- Krishnamoorthy, K., Mathew, T. (2009), Statistical Tolerance Regions: Theory, Applications, and Computation, Wiley.
- Vangel, M. G. (1995), “A Users Guide to RECIPE: A FORTRAN Program for Determining Regression Basis Values,'' http://www.itl.nist.gov/div898/software/recipe/recidoc.pdf

James.M.Womack@aero.org